Astro 201; Project Set #2

due thursday 2/21/2013

1: Stellar Limb Darkening and Exoplanet Transits

The first detection of an extra-solar planet transiting its star was HD 209458b (Charbonneau et al. 2000) – a hot, jupiter-sized planet orbiting $\sim$ 0.05 AU from its host star. The orientation of the system is such that the planet briefly eclipses the star when it passes in front of it, as seen in the original photometric data:

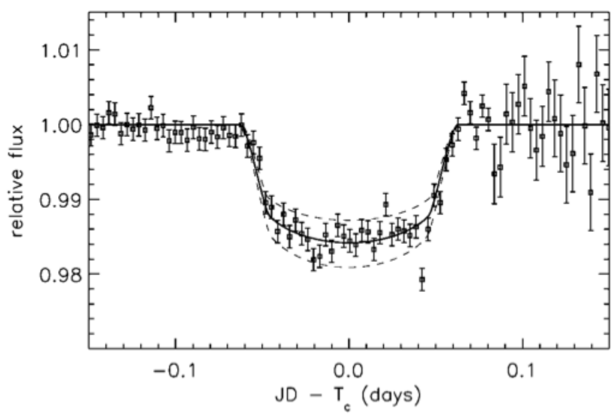

Figure 1: First data on a transiting expolanet, HD209568b, from Charbonneau et al., 2000 . Filled circles show the data binned into 5 minute intervals. The lines show model light curves (which include the limb-darkening effect) a summing different planetary radii.

You'll notice something interesting in the data – the dip in the transit light curve is curved, not flat-bottomed. This is because of the limb-darkening effect – i.e., light emitted from the edge of a star is less intense than light coming from the center. This effect has been known for over a hundred years from observations of the sun (see Figure 2). To accurately determine the properties of a transiting exoplanet (e.g., orbital inclination, planet radius) stellar limb-darkening must be taken into account.

Modeling stellar limb-darkening is one of the classic problems in theoretical astrophysics, going back to Schwarzchild's solution in 1906 and subsequent work by Eddington and Milne.[1] In this project, we'll derive a workable model for exoplanet transit light curves by solving the radiation transport equation (ala Schwarzchild) to account for limb darkening.

[1] Milne 1921 .

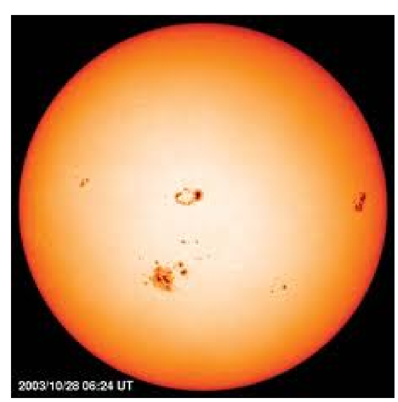

Figure 2: The sun, showing limb darkening (also limb-reddening)

The Limb Darkening Problem

Limb darkening arises because the light from the edge of the star is seen at a slant with respect to the normal vector (see figure 3).

To quantify the effect, we will need to solve the radiation transfer equation to determine the dependence of the specific intensity on the angle ($\mu = \cos\theta$) at which it is observed. The atmospheres of stars like the sun are very thin compared to the radius, so we can use the radiative transfer equation in plane parallel coordinates:

$$\mu \frac{\partial I_\nu(\tau_z, \mu)}{\partial \tau_z} = I_\nu(\tau_z, \mu) - S_\nu(\tau_z, \mu) \tag{1}$$

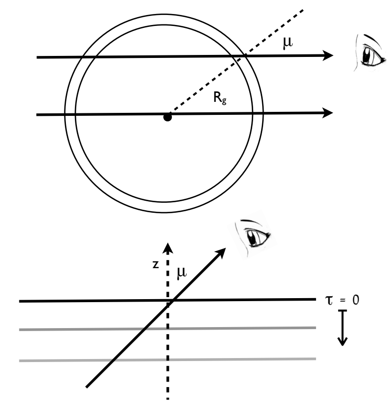

Figure 3: Geometry for the problem. To find the specific intensity observed near the edge of a spherical star (top) we solve the problem for a plane parallel atmosphere (bottom) and find the intensity emerging at the equivalent angle to the normal, $\mu = \cos\theta$.

We might think of this as not just one equation, but an infinite number of differential equations – one for each different frequency ν and direction μ. To make the problem tractable, we'll have to make several typical approximations.

Approximation #1: The opacity is wavelength independent (grey)

In this case, we can drop the ν subscripts in equation 1 and only use quantities that have been integrated over all frequencies. Of course, we can not also drop the μ dependence (i.e., assume the radiation is isotropic) as that would miss the essential physical effect behind limb-darkening. We can simplify life, though, by adopting a relatively simple form for the angular dependence:

Approximation #2: The angular distribution of I can be described by

$$I(\tau_z, \mu) = I_0(\tau_z) + I_1(\tau_z)\mu \tag{2}$$

where we require $I_0 > I_1 > 0$ for all τ_z (so that the intensity never goes negative). This expression is obviously an over-simplification of the angular dependence of I, which could in principle be a complicated function[2] of μ. By specifying this simple form, however, we will no longer need to solve an infinite number of transport equations; instead, we'll only need two equations to determine the two unknowns: I_0 (the isotropic part) and I_1 (the anisotropic part).

[2] At least equation 2 captures one crucial feature: more radiation is moving out of the star ($\mu > 0$) then inward ($\mu < 0$). Our expression actually represents the appropriate limit in the optically thick diffusion regime, as we will discuss later.

a) Write down expressions (in terms of I_0, I_1) for the mean intensity $J(\tau_z)$ and the astrophysical flux $F(\tau_z)$. For fun, also write down the radiation energy density $u(\tau_z)$ and pressure $P(\tau_z)$. What is the ratio P/u and how does it compare to that of isotropic radiation?[3].

[3] The ratio $f = P/u$ is called the Eddington factor and the assumption $f = 1/3$ is called the Eddington approximation.

Now we'll solve the radiation transfer equation. Since there are two unknowns (I_0 and I_1) we'll use two moments of the transfer equation to form a complete system of equations:

b) Take the zeroth moment of the radiation transport equation (i.e., integrate equation 1 over all angles) and show that the source function is given by $S = J$ provided we make another assumption:

Approximation #3: The atmosphere is in radiative equilibrium[4]*, such that the flux,* $F_0 = \sigma T_{\rm eff}^4$, is constant with τ_z.

[4] In the early days, it was not known whether the energy flux in the atmosphere of the sun was carried by radiation or convection. By solving this problem we'll show (as Schwarzchild did in 1906) that the observed limb darkening implies that it the atmosphere is indeed radiative.

c) Take the first moment of the transport equation and use it to find an expression for $I(\tau, \mu)$ which depends on two constants: T_{eff}, and some constant of integration.

To determine the constant of integration, we need to specify some kind of boundary condition. We'll make another approximation:

Approximation #4: At the surface of the star ($\tau = 0$) the inward directed flux (i.e. integrated only over the directions $-1 < \mu < 0$) is zero.

This is one of several approximate boundary conditions that might be chosen. It captures our intuition that the light is streaming out of the stellar surface, but none is streaming inward.

d) Determine the integration constant to complete the solution. Now that you know everything, you should be able to plot the limb darkening law for the relative emergent intensity as a function of μ. Compare your solution to the early observations taken by Muller (1897) shown in Figure 4. Does your calculation indeed suggest that the solar atmosphere is radiative?

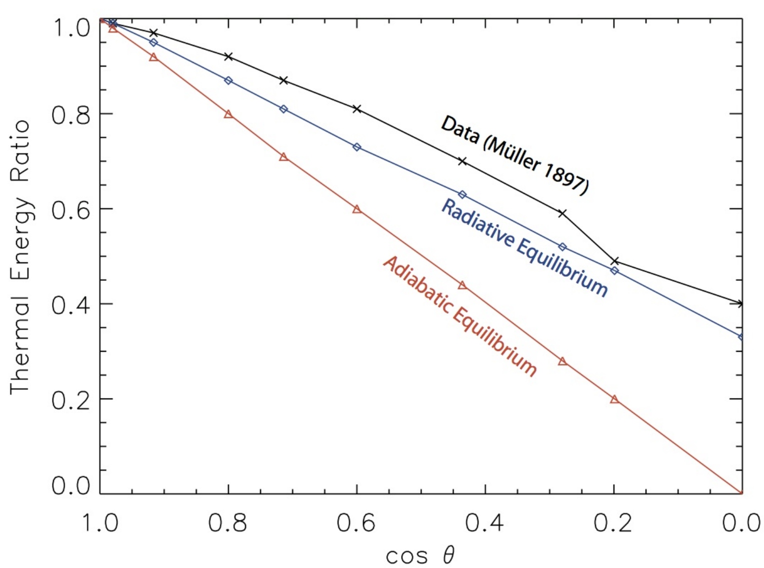

Figure 4: Solutions for the limb darkening law assuming a radiative and a convective atmosphere. Also shown are early observations from Muller (1897). Adapted from K. Schwarzschild (1906) by J. Aufdenberg. The y-axis plots the ratio $I(\tau = 0, \mu)/I(\tau = 0, \mu = 1)$. Note that Schwarzschild made slightly different approximations than we have, and so derived a slightly different solution ($I \propto \mu + 1/2$, shown as the blue line). Your solution will actually fit the data better.

Explaining the Planetary Transit Data

At last we can model (approximately) the light curve of HD209568b. Assume the orbit has an inclination of 90° (i.e., is exactly edge on as viewed from earth) and that the duration of the transit is 0.12 days. For our purposes, you can make the approximation that the radius of the planet R_p is much less than that of the star, $R_\star$ (i.e., no need to integrate over the size of the planet). In this case, you won't get the sharp edges of the light curve quite right, when the planet moves on or off the stellar disk (the ingress and egress marked by w on figure 5).

e) Write an expression for the transit light curve – i.e., the observed flux as a function of time (no need to worry about the overall normalization of the flux; just the relative value as in Figure 1 is fine). Plot your light curve and compare by eye to the observed data in Figure 1.

Comment: You will notice that your light curve has significantly more curvature than the data shown in Figure 1. That is because the inclination of HD209568b is not exactly 90° and the planet does not pass exactly along the mid-plane, rather it grazes its host star, as illustrated in figure 5. **Bonus:** By thinking a little more about the geometry, work out how to use a measurement of the curvature parameter c (defined in figure 5) to determine the inclination of the orbit, based on your equation for limb-darkening (and assuming the stellar and orbital radius are given by other means).

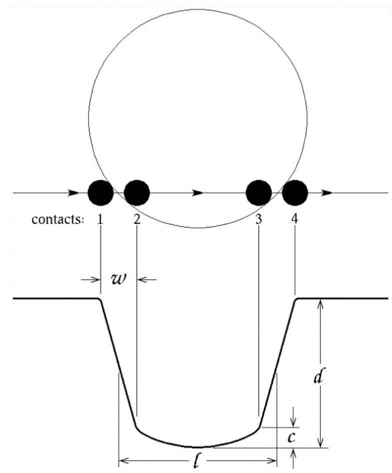

Figure 5: Schematic of the shape of the light curve from an expolanet transit, taken from Brown et al., (2001) .

Explaining the Sun's Red Edge

Notice in Figure 2 that the edge of the sun also looks redder then the center, which is a closely related effect. Let's calculate how much redder. To do so we'll make a further assumption:

Approximation #5: The atmosphere is in LTE, so that the source function $S(\tau)$ is everywhere equal to the blackbody value.

e) Write an expression for the temperature of the atmosphere as a function of τ_z (and depending on $T_{\rm eff}$)[5]. Now assume that when you look at the star at some angle μ, you see roughly a blackbody spectrum at the temperature at a *line of sight* optical depth of $\simeq 2/3$. If the effective temperature of the Sun is $T_{\rm eff} = 5800$ K, how much "cooler" is the spectrum of radiation from the edge ($\mu = 0$) compared to the center ($\mu = 1$)?

[5] Your (approximate) solution for the temperature structure of a radiative atmosphere can be quite a useful one for certain problems.

Comment: If we wanted to derive a more accurate solution for the limb darkening problem, we could write the angular dependence of $I(\tau_z, \mu)$ as an expansion in Legendre polynomials:

$$I(\tau_z, \mu) = \sum_{i=0} I_i(\tau_z) P_i(\mu) \tag{3}$$

What you have done is to solve the problem when keeping only the first two terms in this series. For each additional higher order term you include, a new unknown is introduced and you would need to take another higher moment of the transfer equation to complete the system of equations.[6]

[6] See Chandrasekhar's 1960 book for more. Since the radiation field does not usually oscillate very rapidly with μ, you could probably get away with only the first few terms in the series.

Comment: There are other approximate ways of representing the angular dependence of I. In the *two-stream approximation* we assume that the intensity has one value going up, and another value going down:

$$I(\tau_z, \mu) = I^+(\tau_z)\Theta(0, \pi/2) + I^-(\tau_z)\Theta(\pi/2, \pi) \tag{4}$$

where $\Theta(x_1, x_2) = 1$ for $x_1 < x < x_2$ and 0 otherwise. **Bonus:** If you felt like it, you should be able to use this functional form to solve the transport equation using a similar approach described above.

The generalization of the two stream approximation would be an n-stream approximation:

$$I(\tau_z, \mu) = \sum_i^n I_i \Theta(\mu_i, \mu_i + \Delta\mu_i) \tag{5}$$

Where we would choose the angles μ_i and $\Delta\mu_i$ to give a favorable spacing of the n rays. You should easily convince yourself that $J = \sum_i I_i \Delta\mu_i / 2$. Assuming radiative equilibrium ($J = S$) the set of radiative transfer equations then become:

$$\mu_i \frac{\partial I_i(\tau_z)}{\partial \tau_z} = I_i - \frac{1}{2}\sum_i^n I_i \Delta\mu_i \tag{6}$$

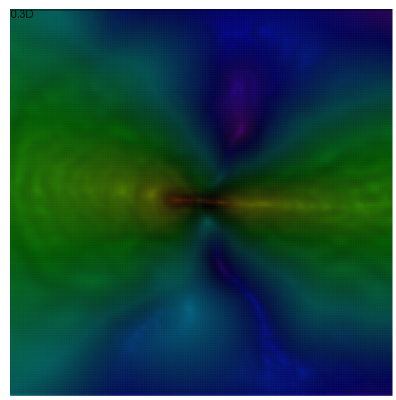

Figure 6: A snapshot of a 3-D hydrodynamical simulation of a dusty torus of gas formed around an AGN. The scale of the image box is $\sim$ 10 parsecs and the BH is an unresolved point at the center. Model from Hopkins and Quataert (2010) ; image rendered by Nathan Roth.

This is just a system of n coupled linear differential equations which in principle can be solved (numerically) to determine the $I_i(\tau_z)$. The higher n, the more accurate the solution. This sort of method of *discrete ordinates* is a common approach to solving radiation transport problems on computers.

1. *Obscured Active Galactic Nuclei*

Accretion onto a supermassive black hole (BH) at the center of a galaxy can power strong radiation – an *active galactic nucleus* (AGN). The BHs powering AGN have masses $\sim 10^7 - 10^8\ M_\odot$ and should radiate mainly in the UV/x-ray. However, in some cases (e.g., the Seyfert 2 AGN) strong emission is seen at around $\sim 10\ \mu$m. It is thought that this infrared radiation is due to the absorption and remission of radiation in dusty gas surrounding the BH. The dusty region is thought to have a spatial extant of $\sim 1 - 10$ pc and a shape like a torus (see figure 1). The origin and properties of AGN dusty torii are an active area of research, with many efforts to model the observed IR spectra with radiation transport codes.

In this project, we develop a simple model of obscured AGN. Although the real systems are clearly aspherical, we'll solve the symmetric analogue – a spherical source surrounded by a spherical envelope.[7]. We'll assume that BH has mass $M_{\rm BH} \sim 10^7\ M_\odot$ and emits a luminosity, $L_{\rm BH}$, equal to it's (electron-scattering) Eddington luminosity. We'll model the source of BH radiation as an isotropically

[7] Our simple setup may be applicable in other astrophysical contexts, e.g., a starburst occurring in a dusty galaxy, a massive star forming in a dusty cloud, or a supernova exploding inside a dusty circumstellar region. One would just need to change the length and luminosity scale.

emitting sphere[8] of radius $R_{\text{in}} \sim 10$ Swarchzhild radii (~ 2 AU). Surrounding the source is a spherical envelope of mainly hydrogen gas with some dust mixed in. We'll take the envelope to have a constant density ρ_0 extending from R_{in} to an outer radius $R_{\text{out}} \sim 10$ parsecs (so $R_{\text{out}} \gg R_{\text{in}}$). We'll assume that radiative heating/cooling dominates the energy exchange in the envelope, so that (given enough time) the envelope will come into radiative equilibrium.

[8] Of course, the emission is actually coming from the accreting material just outside the BH, which is presumably disk-like (not spherical) and does not emit isotropically. But anyway.

The optical depth of the envelope (measured radially, from the center to the edge) is $\tau_0 \simeq \rho_0 \kappa R_{\text{out}}$, where $\kappa \sim 10$ cm^2 g^{-1} is a typical infrared opacity of dusty gas[9], which we will take to be purely absorptive. We'll solve the radiation transport problem separately for the two limits: $\tau_0 \ll 1$ (optically thin) and $\tau_0 \gg 1$ (optically thick). We won't attempt the intermediate case ($\tau_0 \sim 1$) which is actually the hardest to solve, since no simplifying approximation to the transport can be made.

[9] The gas and the dust are typically tightly collisionally coupled (i.e., quickly come into thermal equilibrium with each other) and so we treat them as a single fluid. By opacity, we thus mean the cross-section per unit gram of dusty gas. Most of the mass of this fluid is from hydrogen, so the cross-section of this material is $\sigma \simeq m_p \kappa_{\text{IR}}$ where m_p is the proton mass.

In general, this is a time-dependent problem – i.e., the temperature and density of the envelope is evolving under the influence of gravity[10] and radiation feedback. Here we'll make the *stationarity approximation* – we take a snapshot in time in which the envelope structure is held fixed and solve the steady state radiation transport problem. We'll check the validity of the assumption as we go.

[10] You can safely neglect the self-gravity of the envelope; the gravity of the BH dominates.

I. Optically thin case

In the optically thin limit, most photons free-stream through the envelope without interacting, and the radiation field can be approximated by the value it would have in empty space[11]. We take the opacity to be independent of wavelength.

[11] It is true that a fraction $\sim \tau_0$ of the photons are absorbed in the envelope, and so the radiation field is not exactly the same as it would be in empty space, but let's not worry about that small fraction.

a) Assume that the specific intensity from the surface of our spherical source is given by Planck's function at a temperature T_s. What is T_s? At around what wavelengths does the BH radiate?

b) Solve for the temperature profile, $T(r)$ (in terms of T_s and R_{in}) of the dusty envelope assuming that it is in radiative equilibrium. What is a characteristic temperature of the envelope?[12] At around what wavelengths does the envelope radiate?

[12] Most of the mass of the envelope is at large radii, so you might evaluate $T(r)$ at, say, $r \sim R_{\text{out}}/2$.

c) Let's check that our use of the stationarity approximation is reasonable. What is the characteristic time scale, t_{esc} for photons to escape the dusty envelope? How does this compare to the dynamical timescale, t_{dyn} (e.g., the time it would take to outer edge of the envelope to free-fall into the black hole)? Also check that the timescale, t_{eq}, for the envelope to come into radiative equilibrium is short compared to the dynamical timescale.[13] Does it seem safe to assume

[13] As a rough test, it will suffice to look at the value of either the cooling or heating time for a characteristic equilibrium temperature at, say, $R_{\text{out}}/2$.

the envelope structure is fixed when solving this radiation transport problem?

II. Optically thick case

Next consider the opposite limit, in which the envelope is optically thick and the diffusion approximation applies. Continue to assume the opacity is grey and purely absorptive. The diffusion equation in a general coordinate system is

$$\mathbf{F} = -\frac{c}{3\kappa\rho}\nabla u(r) \tag{7}$$

where $\mathbf{F}$ is the flux vector. If we assume that radiative equilibrium holds, it follows from the zeroth moment of the radiative transfer equation that

$$\nabla \cdot \mathbf{F} = 0 \tag{8}$$

d) In plane parallel coordinates, the condition of radiative equilibrium requires that the flux be constant. Show that in spherical coordinates the assumption of radiative equilibrium requires instead that the luminosity, $L = 4\pi r^2 F$, is constant with radius.

e) What is the expression for the characteristic time scale, $t_{\rm esc}$ for photons to escape the dusty envelope in this optically thick case? At about what value of τ_0 does our stationarity approximation become questionable?

f) Solve the diffusion equation to determine the temperature profile of the dusty envelope $T(r)$ in this optically thick case. You'll need to specify a boundary condition – we'll take it to be that the temperature is zero at $R_{\rm rout}$, the so-called "radiative zero" boundary condition.[14]

[14] This is not necessarily the best boundary condition. A better one might be that $T(R_{\rm out}) = T_{\rm eff}$, where we define the effective temperature by $L_{\rm BH} = 4\pi R_{\rm out}^2 \sigma_{\rm sb} T_{\rm eff}^4$.

Bonus (optional): Our assumption of a constant density envelope is a little unrealistic. However, you can easily solve the same diffusion problem using a power-law density profile $\rho(r) = \rho_0 (r/r_{\rm in})^{-\zeta}$.

g) Plot the two temperature profiles you have derived (optically thin and optically thick cases) in comparison to each other. For the optically thick case, take $\tau_0 = 100$. For the optically thin case take $\tau_0 = 0.1$. Note that dust is sublimated (destroyed) at temperatures higher than ~ 1500 K. Within about what radii do expect dust destruction to be important?

h): Consider the dusty gas at about the middle of the envelope ($r = R_{\rm out}/2$). How does the temperature you find for the optically thick case compare to that of the optically thin case? Argue why the ratio of these temperatures makes physical sense.

III. Optically thick, non-grey case

i): For real dust, the opacity is not grey. For wavelengths $\lambda > 1\ \mu$m, the wavelength dependence is roughly a power law, $\kappa = \kappa_0(\lambda/\lambda_0)^{-n}$, with $n \approx 2$. Show that in this case the Rosseland mean opacity $\kappa_R(T) \propto T^n$ (you don't need bother to find the constants). Explain why it makes sense that the mean opacity increases with T.

j): Take the Rosseland mean opacity to be $\kappa_R = \kappa_{R,0}(T/T_0)^2$, where $\kappa_{R,0} = 10\ \mathrm{cm}^2\ \mathrm{g}^{-1}$ and $T_0 = 1000$ K. Find an expression for the temperature structure in the non-grey, optically thick case.

Comment: Our model is too simple to apply to real obscured AGN, for many reasons: (1) We have assumed a spherical geometry, when in reality the envelope is probably a torus; (2) We have assumed the density distribution is smooth and uniform, while more detailed models suggest that it is highly clumpy. Nevertheless, our analytic solutions may be useful for testing full blown radiation transfer codes. They may also provide some intuition into the results of detailed 3-D calculations (e.g., the optically thin solution might best correspond to the polar region, where the gas column densities are lower, while the optically thick solution might better correspond to the dense equatorial regions.